Quantisation of presymplectic manifolds and principal series representations of complex-semisimple Lie groups

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Abstract

We generalise the 'quantisation commutes with induction' result we used earlier in connection with discrete series representations, to a principle applicable in more general settings. This requires the use of presymplectic manifolds (manifolds equipped with a closed two-form) instead of symplectic ones. Using this principle and Penington and Plymens's proof of the Connes–Kasparov conjecture for complex-semisimple Lie groups, we prove that quantisation commutes with reduction at the connected components of the principal series of a complex-semisimple Lie group. As an application, we realise the K-theory classes associated to these connected components as quantisations of fibre bundles over the associated coadjoint orbits.

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Introduction

Let G be Lie group acting on a symplectic manifold (M, ω) in Hamiltonian fashion, and let V be an irreducible representation of G associated to (the coadjoint orbit through) an element $\xi \in \mathfrak{g}^*$, the dual of the Lie algebra of G. Then the *quantisation commutes with reduction* principle states that

$$R_G^V \circ Q_G(M, \omega) = Q(M_\xi, \omega_\xi),$$

where Q_G and Q denote geometric quantisation, and the quantum reduction map R_G^V is defined by taking multiplicities of V. Furthermore, (M_ξ, ω_ξ) is the Marsden–Weinstein reduction [12] of (M, ω) at ξ, i.e. $M_\xi = \Phi^{-1}(\xi)/G_\xi$, with $\Phi : M \to \mathfrak{g}^*$ a momentum map for the action. If M and G are compact, this principle has been given explicit meaning, and been proved, by several authors [14, 15, 16, 22]. The geometric quantisation of (M, ω) is then defined as the equivariant index of a Dirac operator $D\!\!\!\!/\,_M^L$ on M, coupled to a line bundle $L \to M$ with Chern class $[\omega]$:

$$Q_G(M, \omega) = G\text{-index}\left(D\!\!\!\!/\,_M^L\right). \tag{1}$$

For M and/or G noncompact, results have been achieved in two directions. For compact G and noncompact M, there are results by Paradan [18, 19, 20], and by Ma and Zhang [13]. If M and G are both allowed to be noncompact, but the orbit space M/G is still compact, Landsman [9] has proposed a definition based on the analytic assembly map μ_M^G used in the Baum–Connes conjecture [1]:

$$Q_G(M, \omega) = \mu_M^G \left[D\!\!\!\!/\,_M^L\right] \in K_0(C^*_{(r)}(G)). \tag{2}$$

Here $\left[D\!\!\!\!/\,_M^L\right]$ denotes the K-homology class of the Dirac operator $D\!\!\!\!/\,_M^L$ on M, and $K_0(C^*_{(r)}(G))$ is the K-theory group of the (full or reduced) group C^*-algebra of G. This definition reduces to (1) in the compact case. Results based on this definition have been achieved by Landsman, Hochs, Mathai and Zhang [6, 7, 11].

In [7], we proved a result about reduction at discrete series representations of real-semisimple Lie groups, using Landsman's definition of quantisation. This result was based on a *quantisation commutes with induction* principle, which allowed us to deduce our result from the compact case. In the present paper, we apply the same principle to reduction at (families of) principal series representations of complex-semisimple Lie groups. This application is based on Penington and Plymen's proof of the Connes-Kasparov conjecture [21], in which they give an explicit description of the K-theory of the reduced C^*-algebra of a complex-semisimple Lie group.

The key assumption in [7] that made the Hamiltonian induction construction used there possible, is that the momentum map of the action in question takes values in the strongly elliptic set $\mathfrak{g}^*_{\mathrm{se}} \subset \mathfrak{g}^*$ of elements with compact stabilisers under the coadjoint action. This set is empty for complex-semisimple groups, so that we will lose some desirable properties of Hamiltonian induction. The most important of these is nondegeneracy of induced symplectic forms. We will

therefore consider quantisation of *presymplectic manifolds.* To us, a presymplectic manifold is a smooth manifold equipped with a closed two-form. Results on presymplectic manifolds and their quantisations have been obtained by Cannas da Silva, Karshon and Tolman [2, 8]. Since the presymplectic manifolds we consider may be odd-dimensional, the quantisation of these spaces may end up in *odd* K-theory. Then the even K-theory group $K_0(C^*_{(r)}(G))$ in (2) should be replaced by $K_1(C^*_{(r)}(G))$.

For representations that are not isolated in the unitary dual of a group, the meaning of a multiplicity is less clear. For this reason, we will consider reduction at *connected components* of the principal series of complex-semisimple groups, i.e. families of principal series representations, which turns out to be well-defined on the level of K-theory. As an application of our result, we show how the K-theory class associated to such a family can be obtained as the quantisation of a *fibre bundle* over the associated coadjoint orbit. This coadjoint orbit is a symplectic manifold, but the pullback of the symplectic form to the total space of this fibre bundle is only presymplectic, and our result on presymplectic quantisation applies.

Statement of the results

Let G be a complex-semisimple Lie group, with a maximal compact subgroup $K < G$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition at the Lie algebra level. Let (N, ν) be a compact, Spin^c-prequantisable Hamiltonian K-manifold, with momentum map $\Phi^N : N \to \mathfrak{k}^*$. Form the manifold $M := G \times_K N$ as the quotient of $G \times N$ by the free K-action given by $k \cdot (g, n) = (gk^{-1}, kn)$, for $k \in K$, $g \in G$ and $n \in N$. Consider the G-invariant two-form ω on M defined as follows. For $n \in N$, $v, w \in T_nN$ and $X, Y \in \mathfrak{p}$, we have the tangent vectors $Tp(X + v)$ and $Tp(Y + w)$ to M at $[e, n]$, where $p : G \times N \to M$ is the quotient map. Then ω is defined by the properties that it is G-invariant, and

$$\omega_{[e,n]}(X + v, Y + w) := \nu_n(v, w) - \langle \Phi^N(n), [X, Y] \rangle.$$

Furthermore, we will see in Section 1.1 that the K-theory group $K_d(C^*_r(G))$, for $d = \dim(G/K)$, is generated by elements $[\pi_{\lambda+\rho}]$, where λ runs over the set Λ^*_+ of dominant weights for $\mathfrak{k}$, with respect to some maximal torus and a choice of positive roots. The class $[\pi_{\lambda+\rho}] \in K_d(C^*_r(G))$ is associated to the connected component of the principal series of G corresponding to the parameter $\lambda + \rho$. Our main result is the following.

Theorem 0.1 (Quantisation commutes with reduction at families of principal series representations)**.** *The pair (M, ω) is an equivariantly* Spin^c*-prequantisable presymplectic manifold, and the action of G on M is pre-Hamiltonian. The* Spin^c*-quantisation of this action decomposes as*

$$Q_G(M, \omega) = \sum_{\lambda = i\xi \in \Lambda^*_+} Q(N_\xi, \nu_\xi)[\pi_{\lambda+\rho}].$$

This is the first result on K-theoretic quantisation in which an explicit decomposition of the quantised space is obtained.

As an application of Theorem 0.1, we obtain an 'orbit method' for the families $[\pi_{\lambda+\rho}]$ of principal series representations of G. Because these are, in a sense, *bundles* of irreducible representations, we will obtain them as quantisations of bundles over coadjoint orbits.

More explicitly, let $\lambda = i\xi \in \Lambda_+^*$ be given. Let $\mathcal{O}^\lambda \subset \mathfrak{g}^*$ be the coadjoint orbit through ξ, and let ω^λ be the standard symplectic form on $\mathcal{O}^\lambda$. Consider the manifold

$$M^\lambda := G \times_K (K \cdot \xi),$$

which is constructed as above, for $N = K\cdot\xi$, the K-coadjoint orbit of ξ. Consider the map

$$\pi : M^\lambda \to \mathcal{O}^\lambda \tag{3}$$

defined by

$$\pi[g, k\xi] = gk\xi,$$

and the two-form $\pi^*\omega^\lambda$ on M^λ. Note that this form is closed since ω^λ is, so that $(M^\lambda, \pi^*\omega^\lambda)$ is a presymplectic manifold.

Corollary 0.2 (Orbit method for families of principal series representations)**.** *The action of G on the presymplectic manifold $(M^\lambda, \pi^*\omega^\lambda)$ has the following properties.*

1. *The map π defines an equivariant fibre bundle over $\mathcal{O}^\lambda$, with fibres diffeomorphic to G_ξ/K_ξ.*
2. *The G-action on $(M^\lambda, \pi^*\omega^\lambda)$ is equivariantly* Spin^c*-prequantisable.*
3. *The quantisation of this action is the class $[\pi_{\lambda+\rho}] \in K_d(C_r^*(G))$.*

Note that the fibre G_ξ/K_ξ of the bundle π is contractible. If λ is a *regular* weight, then we have $G_\xi = MA$ and $K_\xi = M$, so that this fibre equals the subgroup $A < G$ in an Iwasawa decomposition $G = KAN$. Then this fibre is homeomorphic to $i\mathfrak{a}^*$, i.e. to the connected component of the principal series of G associated to the class $[\pi_{\lambda+\rho}]$. Note that in the case considered in [7], we had $G_\xi = K_\xi$, so that M^λ is equal to the coadjoint orbit $\mathcal{O}^\lambda$.

Notation

We will write d_X for the dimension of a manifold X. In particular, we will write $d := d_{G/K}$. Where appropriate, these dimensions should be interpreted modulo 2.

1 The principal series and K-theory

In [21], Penington and Plymen prove the Connes–Kasparov conjecture for complex semisimple Lie groups, via a very explicit description of the K-theory of the C^*-algebra of such groups in terms of their principal series representations. We briefly recall the parts of their work we will use. For details and proofs, we refer to [21], and references therein.

1.1 The K-theory of $C^*_r(G)$

Let G be a complex semisimple Lie group. Let $G = KAN$ be an Iwasawa decomposition, with $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ at the Lie algebra level. Then $M := Z_K(\mathfrak{a})$ is a Cartan subgroup of K, with Lie algebra $\mathfrak{m}$, and MA is a Cartan subgroup of G. Fix a set of positive roots R^+ for $(\mathfrak{g}, \mathfrak{m} \oplus \mathfrak{a})$, and let Λ^*_+ be the corresponding set of dominant weights for $(\mathfrak{k}, \mathfrak{m})$. Consider the Weyl group

$$W := N_K(M)/M.$$

For $\lambda \in \Lambda^*_+$, let W_λ be the stabiliser of λ in W. The the principal series representations of G are parametrised by the set

$$\hat{G}_{\mathrm{ps}} = \bigcup_{\lambda \in \Lambda^*_+} E_\lambda, \tag{4}$$

where

$$E_\lambda := \{\lambda\} \times i\mathfrak{a}^*/W_\lambda$$

is a connected component of $\hat{G}_{\mathrm{ps}}$. (Here the action of W on $i\mathfrak{a}^*$ is defined via the isomorphism $\mathfrak{a} = i\mathfrak{m}$.)

The reduced group C^*-algebra $C^*_r(G)$ of G can be described very explicitly in terms of principal series representations. There is a bundle of Hilbert spaces

$$\mathcal{H} = (\mathcal{H}_\lambda)_{\lambda \in \Lambda^*_+} \to \hat{G}_{\mathrm{ps}},$$

the fibres of which only depend on the discrete parameter λ. That is, the fibre is constant on every connected component E_λ of $\hat{G}_{\mathrm{ps}}$. Now $C^*_r(G)$ is isomorphic to the continuous sections, vanishing at infinity, of the bundle $\mathcal{K}(\mathcal{H}) \to \hat{G}_{\mathrm{ps}}$ of compact operators on the fibres of $\mathcal{H}$. Since the fibres of $\mathcal{H}$ only depend on λ, we obtain

$$K_*(C^*_r(G)) \cong K^*(\hat{G}_{\mathrm{ps}}), \tag{5}$$

the topological K-theory of $\hat{G}_{\mathrm{ps}}$.

To compute $K^*(\hat{G}_{\mathrm{ps}})$, we distinguish two kinds of λ:

1. λ regular, i.e. $W_\lambda = \{e\}$.
2. λ singular, i.e. $W_\lambda \neq \{e\}$.

The set of regular elements of Λ^*_+ is exactly equal to $\Lambda^*_+ + \rho$, with ρ half the sum of the positive roots of G. In the decomposition (4), a connected component E_λ equals $\{\lambda\} \times i\mathfrak{a}^*$ if λ is regular, and is homeomorphic to a closed half space in $i\mathfrak{a}^*$ if λ is singular. Therefore, one has

$$K^j(E_\lambda) = 0$$

for singular λ, whereas for regular λ, we get

$$K^j(E_\lambda) = \begin{cases} \mathbb{Z} & \text{if } j = d; \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\dim(\mathfrak{a}) = d$ modulo 2, since $\mathfrak{n}$ is a sum of root spaces of real dimension 2.

So explicitly, we have for regular $\lambda \in \Lambda^*_+$,

$$K^d(E_\lambda) = \mathbb{Z} \cdot [\pi_\lambda],$$

where $[\pi_\lambda] \in K^d(i\mathfrak{a}^*)$ is the Bott generator. We conclude that

$$\begin{aligned} K_d(C^*_r(G)) &= \bigoplus_{\lambda \in \Lambda^*_+} \mathbb{Z} \cdot [\pi_{\lambda+\rho}]; \\ K_{d+1}(C^*_r(G)) &= 0. \end{aligned} \tag{6}$$

We will interpret the K-theory class $[\pi_{\lambda+\rho}] \in K_*(C^*_r(G))$ as the generator associated with the family of principal series representations associated to the connected component $E_{\lambda+\rho}$ of $\hat{G}_{\mathrm{ps}}$. As noted in Corollary 0.2, the class $[\pi_{\lambda+\rho}]$ can be described as the geometric quantisation of a certain fibre bundle over the coadjoint orbit through λ.

We define the reduction map at the connected component $E_{\lambda+\rho}$ of $\hat{G}_{\mathrm{ps}}$ as the multiplicity of $[\pi_{\lambda+\rho}]$.

Definition 1.1. For $\lambda \in \Lambda^*_+$, the *reduction map at* $E_{\lambda+\rho}$,

$$R^{\lambda+\rho}_G : K_*(C^*_r(G)) \to \mathbb{Z},$$

is defined by

$$R^{\lambda+\rho}_G \left(\sum_{\lambda' \in \Lambda^*_+} m_{\lambda'+\rho} [\pi_{\lambda+\rho}] \right) = m_{\lambda+\rho}.$$

A crucial step in Penington and Plymen's proof of the Connes–Kasparov conjecture is the fact that the reduction map $R^{\lambda+\rho}_G$ and the usual reduction map for the compact group K are directly related to each other via *Dirac induction.*

1.2 Dirac induction

Dirac induction is a map

$$\text{D-Ind}^G_K : R(K) \to K_*(C^*_r(G)),$$

which is defined in terms of Dirac operators $\not{D}^V$ on G/K, coupled to given irreducible representations V of K. For this map to be well-defined, we will assume that the representation $\mathrm{Ad} : K \to \mathrm{SO}(\mathfrak{p})$ lifts to $\widetilde{\mathrm{Ad}} : K \to \mathrm{Spin}(\mathfrak{p})$. It may be necessary to replace G and K by double covers for this lift to exist, although Penington and Plymen describe how to handle the case where a lift $\widetilde{\mathrm{Ad}}$ does not exist in Section 6 of [21].

Let Δ_d be the standard representation of $\mathrm{Spin}(d)$. Dirac induction is defined by

$$\text{D-Ind}_K^G[V] := \Big[\big(C_r^*(G) \otimes \Delta_d \otimes V\big)^K, b(\not{D}^V)\Big] \\ \in KK_*(\mathbb{C}, C_r^*(G)) \cong K_*(C_r^*(G)), \quad (7)$$

where, for an orthonormal basis $\{X_j\}$ of $\mathfrak{p}$,

$$\not{D}^V := \sum_{j=1}^{d} X_j \otimes c(X_j) \otimes 1_V \quad (8)$$

is the Spin-Dirac operator on G/K, and $b : \mathbb{R} \to \mathbb{R}$ is a normalising function, e.g. $b(x) = \frac{x}{\sqrt{1+x^2}}$. Here in the first factor, X_j is viewed as a vector field on G, and in the second factor, c denotes the Clifford action.

If $\mathfrak{p}$ is even-dimensional, then Δ_d splits into two irreducibles: $\Delta_d = \Delta_d^+ \oplus \Delta_d^-$. The Dirac operator (8) is odd with respect to the induced grading on $\mathcal{E}_V$. Therefore, Dirac induction takes values in $KK_d(\mathbb{C}, C_r^*(G))$.

The Connes–Kasparov conjecture states that that the Dirac induction map is an isomorphism of abelian groups. This was proved for complex-semisimple Lie groups G by Penington and Plymen in [21]. In [23], Wassermann proves the Connes–Kasparov conjecture for linear reductive groups.

The reduction map for K at a dominant weight $\lambda \in \Lambda_+^*$,

$$R_K^\lambda : R(K) \to \mathbb{Z},$$

with $R(K)$ the representation ring of K, is defined by taking the multiplicity of the irreducible K-representation V_λ with highest weight λ:

$$R_K^\lambda([V]) := [V : V_\lambda],$$

for any finite-dimensional representation V of K. The key step in Penington and Plymen's proof of the Connes–Kasparov conjecture is the following relation between the reduction maps for G and K, and the Dirac induction map.

Proposition 1.2. *The following diagram commutes for every $\lambda \in \Lambda_+^*$:*

$$\begin{tikzcd} K_*(C_r^*(G)) \arrow[dr, "R_G^{\lambda+\rho}"] & \\ R(K) \arrow[u, "\text{D-Ind}_K^G"] \arrow[r, "R_K^\lambda"'] & \mathbb{Z}. \end{tikzcd}$$

This result is proved in Section 5 of [21]. It will allow us to deduce Theorem 0.1 from the compact case, by using the notion of pre-Hamiltonian induction, which will be introduced in the next section.

2 Pre-Hamiltonian induction and quantisation

In this section, we consider any connected Lie group G with a maximal compact subgroup $K < G$, and an $\mathrm{Ad}(K)$-invariant subspace $\mathfrak{p} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We will of course later apply this to complex semisimple groups G. We equip $\mathfrak{g}$ with any $\mathrm{Ad}(K)$-invariant inner product with respect to which $\mathfrak{k} \perp \mathfrak{p}$. As in Subsection 1.2, we assume that the representation $\mathrm{Ad} : K \to \mathrm{SO}(\mathfrak{p})$ lifts to $\widetilde{\mathrm{Ad}} : K \to \mathrm{Spin}(\mathfrak{p})$, so that G/K is a Spin manifold.

2.1 Pre-Hamiltonian induction

In [7], the notion of *Hamiltonian induction* was introduced, which assigns a Hamiltonian G-manifold $M = G \times_K N$ with an equivariant prequantisation to a prequantised Hamiltonian K-manifold N. There is an inverse construction called *Hamiltonian cross-section.* A crucial assumption in [7] was that the momentum map $\Phi^M : M \to \mathfrak{g}^*$ takes values in the strongly elliptic set $\mathfrak{g}^*_{\mathrm{se}}$, i.e. the set of elements of $\mathfrak{g}^*$ with compact stabilisers under the coadjoint action. We will apply this construction to complex semisimple groups G, for which $\mathfrak{g}^*_{\mathrm{se}}$ is empty. Therefore, we drop the assumption that $\Phi^M(M) \subset \mathfrak{g}^*_{\mathrm{se}}$. This has two main consequences for the Hamiltonian induction process:

- the two-form on M induced by the symplectic form on N may be degenerate;
- Hamiltonian cross-sections are no longer well-defined (specifically, the subsets $N \subset M$ taken in this process may not be smooth submanifolds).

Because of the latter point, we will not be able to use Hamiltonian cross-sections. The first point means we will have to deal with *presymplectic manifolds.* There are different variations of the definition of presymplectic manifolds (sometimes the rank of the two-form considered is assumed to be constant). We will simply consider manifolds with closed two-forms on them.

Definition 2.1. A *presymplectic form* on a smooth manifold M is a closed two-form $\omega \in \Omega^2(M)$.

Note that in the more general case we now consider, without reference to the strongly elliptic set, the manifold G/K may be odd-dimensional. For a Hamiltonian K-manifold N, the induced manifold $M = G \times_K N$ is then odd-dimensional, so that it can never be symplectic.

The definition of a momentum map for an action by a Lie group on a presymplectic manifold is completely analogous to the symplectic case. If such a map

exists, we call the action *pre-Hamiltonian.* A prequantisation of a presymplectic manifold (M, ω) is also defined as in the symplectic case. In particular, we will use Spin^c-prequantisation, which involves a line bundle $L^{2\omega} \to M$ with a connection whose curvature is $2\pi i \cdot 2\omega$, and a Spin^c-structure[1] $P^M \to M$ with determinant line bundle $L^{2\omega}$.

In the presymplectic setting, we will call the process analogous to Hamiltonian induction *pre-Hamiltonian induction.* It is defined completely analogous to Hamiltonian induction, as described in Sections 2 and 3 of [7]. We will briefly review the constructions here.

For any Lie group H, let pHamPS(H) be the set of pre-Hamiltonian H-actions with equivariant Spin^c-prequantisations and Spin^c-structures, which consists of classes of septuples

$$\big(M, \omega, \Phi^M, L^{2\omega}, (-,-)_{L^{2\omega}}, \nabla^M, P^M\big),$$

where

- (M, ω) is a presymplectic manifold, equipped with an H-action that preserves ω;
- $\Phi^M : M \to \mathfrak{h}^*$ is a momentum map for this action;
- $\big(L^{2\omega}, (-,-)_{L^{2\omega}}, \nabla^M\big)$ is an H-equivariant Spin^c-prequantisation of (M, ω);
- $P^M \to M$ defines an H-equivariant Spin^c-structure on M, with determinant line bundle $L^{2\omega}$, such that M is complete in the Riemannian metric induced by P^M.

Two of such septuples are identified if there is an equivariant diffeomorphism between the manifolds in question, which relates the presymplectic forms, momentum maps, line bundles and metrics on them via pullback.[2]

We will also use the sets

- CpHamPS(H), for which it is in addition assumed that the orbit space M/H is compact;
- CHamPS(H), for which orbit spaces should be compact, and ω should be an actual symplectic form; and
- ECpHamPS(H), consisting of all classes in CpHamPS(H) for which M is even-dimensional.

[1] We will slightly abuse terminology, by using the term Spin^c-structure on a manifold X for a principal Spin^c-bundle $P \to X$ such that $P \times_{\mathrm{Spin}^c(d_X)} \mathbb{R}^{d_X} \cong TX$, without explicitly referring to this isomorphism.

[2] We do not explicitly require that such a diffeomorphism relates the connections and Spin^c-structures of two such septuples to each other. All that is needed for the purposes of geometric quantisation is that it relates the curvatures of connections, and the determinant line bundles of Spin^c-structures, and this follows from the other properties.

Definition 2.2. *Pre-Hamiltonian induction* is the map[3]

$$\text{pH-Ind}_K^G : \text{pHamPS}(K) \to \text{pHamPS}(G),$$

given by

$$\text{pH-Ind}_K^G \big[N, \nu, \Phi^N, L^{2\nu}, (-,-)_{L^{2\nu}}, \nabla^N, P^N\big] = \big[M, \omega, \Phi^M, L^{2\omega}, (-,-)_{L^{2\omega}}, \nabla^M, P^M\big], \quad (9)$$

as defined below.

- The manifold $M = G \times_K N$ is the quotient of $G \times N$ by the K-action defined by $k(g,n) = (gk^{-1}, kn)$, for $k \in K$, $g \in G$ and $n \in N$.
- The G-invariant two-form $\omega \in \Omega^2(M)$ is defined by

  $$\omega_{[e,n]}(v + X, w + Y) = \nu_n(v,w) - \langle \Phi^N(n), [X,Y] \rangle, \quad (10)$$

  for $n \in N$, $v, w \in T_nN$ and $X, Y \in \mathfrak{p}$, where we note that $T_{[e,n]}M \cong T_nN \oplus \mathfrak{p}$.
- The momentum map $\Phi^M : M \to \mathfrak{g}^*$ is defined by

  $$\Phi^M([g,n]) = \text{Ad}^*(g)\Phi^N(n),$$

  for $g \in G$ and $n \in N$.
- The line bundle $L^{2\omega}$ equals $G \times_K L^{2\nu} \to M$.
- The Hermitian metric $(-,-)_{L^{2\omega}}$ on $L^{2\omega}$ is given by

  $$\big([g,l],[g',l']\big)_{L^{2\omega}} = (l,l')_{L^{2\nu}},$$

  for $g, g' \in G$, $n \in N$ and $l, l' \in L^{2\nu}_N$.

The definitions of the connection ∇^M on $L^{2\omega}$ and the Spin^c-structure P^M on M are a little more involved, and we refer to Section 3 of [7] for details.

We then have the following result, which can be proved completely analogously to Sections 2 and 3 in [7], omitting the assumption that momentum maps take values in strongly elliptic sets.

Theorem 2.3 (pre-Hamiltonian induction)**.** *Pre-Hamiltonian induction is well-defined, in the sense that (M,ω) is a presymplectic manifold, Φ^M is a momentum map, $\big(L^{2\omega}, (-,-)_{L^{2\omega}}, \nabla^M\big)$ is a G-equivariant prequantisation of (M,ω), and $P^M \to M$ defines a G-equivariant* Spin^c*-structure on M, with determinant line bundle $L^{2\omega}$.*

[3] The author wishes to point out that the occurrence of his initials in the symbol for the pre-Hamiltonian induction map is purely coincidental.

2.2 Quantisation commutes with induction

As in [6, 7, 9], we define geometric quantisation as the analytic assembly map used in the Baum–Connes conjecture [1] applied to the classes defined by Dirac operators in K-homology [5].

Consider two classes

$$\begin{aligned}\big[N, \nu, \Phi^N, L^{2\nu}, (-,-)_{L^{2\nu}}, \nabla^N, P^N\big] &\in \mathrm{CpHamPS}(K);\\ \big[M, \omega, \Phi^M, L^{2\omega}, (-,-)_{L^{2\omega}}, \nabla^M, P^M\big] &\in \mathrm{CpHamPS}(G).\end{aligned}$$

Then one has Spin^c-Dirac operators [3, 4, 10] $D\!\!\!/_N^{L^{2\nu}}$ and $D\!\!\!/_M^{L^{2\omega}}$ on N and M, respectively, coupled to the line bundles in question via the given connections. These Dirac operators define K-homology classes

$$\begin{aligned}\Big[D\!\!\!/_N^{L^{2\nu}}\Big] &\in K^K_{d_N}(N);\\ \Big[D\!\!\!/_M^{L^{2\omega}}\Big] &\in K^G_{d_M}(M).\end{aligned}$$

Definition 2.4. The *quantisation maps*

$$\begin{aligned}Q_K &: \mathrm{CpHamPS}(K) \to K_{d_N}(C^*K);\\ Q_G &: \mathrm{CpHamPS}(G) \to K_{d_M}(C^*_rG),\end{aligned}$$

are defined by

$$\begin{aligned}Q_K\big[N, \nu, \Phi^N, L^{2\nu}, (-,-)_{L^{2\nu}}, \nabla^N, P^N\big] &= \mu^K_N\Big[D\!\!\!/_N^{L^{2\nu}}\Big];\\ Q_G\big[M, \omega, \Phi^M, (-,-)_{L^{2\omega}}, \nabla^M, P^M\big] &= \mu^G_M\Big[D\!\!\!/_M^{L^{2\omega}}\Big],\end{aligned}$$

where the maps μ^K_N and μ^G_M are analytic assembly maps.

Restricting to even-dimensional manifolds N, and noting that $\mu^K_N : K^K_0(N) \to K_0(C^*K)$ then equals the usual equivariant index

$$K\text{-index} : K^K_0(N) \to R(K),$$

we have the following result.

Theorem 2.5 (Quantisation commutes with pre-Hamiltonian induction)**.** *The following diagram commutes:*

$$\begin{tikzcd} \mathrm{CpHamPS}(G) \arrow[r, "Q_G"] & K_d(C^*_r(G)) \\ \mathrm{ECpHamPS}(K) \arrow[u, "\text{pH-Ind}^G_K"] \arrow[r, "Q_K"] & R(K), \arrow[u, "\text{D-Ind}^G_K"'] \end{tikzcd}$$

where D-Ind^G_K *is the Dirac induction map* (7)*.*

The proof of this theorem is completely analogous to the proof of Theorem 4.5 in [7]. In the proof of that theorem, Corollary 3.13 and Theorem A.1 from [6] are used. The proofs of these results are given for the even-dimensional, graded case, and only the even parts of K-homology and K-theory are used. The facts that the Hilbert spaces and modules are graded, and the operators on them are odd, are not used specifically, however, so that the arguments carry through to the odd case.

The parity of the dimension d of G/K, and hence of $G \times_K N$, for N even-dimensional, mainly comes into play in the following sub-diagram of Diagram (28) in [7]:

$$\begin{tikzcd} K_d^{G\times K\times K}(G\times N) \arrow[r, "\mu_{G\times N}^{G\times K\times K}"] & K_d(C_r^*(G\times K\times K)) \\ K_0^K(N) \arrow[u, "{[\not{D}_{G,K}]\times -}"] \arrow[r, "\mu_N^K"] & R(K). \arrow[u, "{\mu_G^{G\times K}[\not{D}_{G,K}]\times -}"'] \end{tikzcd} \tag{11}$$

Here the class

$$[\not{D}_{G,K}] \in K_d^{G\times K}(G) \tag{12}$$

is the class defined by the elliptic operator on the trivial bundle $G \times \Delta_d \to G$ defined by the formula (8), for $V = \mathbb{C}$ the trivial representation. In [7], d was even, so only even K-homology and K-theory appeared in Diagram (11). In general, taking the Kasparov product with the class (12), or its image $\mu_G^{G\times K}[\not{D}_{G,K}]$ under the assembly map, preserves degrees in K-homology and K-theory if G/K is even-dimensional, and changes degrees between even and odd if it is odd-dimensional. This is the only place in the proof of Theorem 2.5 where the dimension of G/K, and hence the degree in K-homology and K-theory plays an explicit role. The rest of the proof is the same as the proof of Theorem 4.5 in [7].

3 Proofs of the results

We now specialise to the case where G is a complex-semisimple Lie group again. We prove Theorem 0.1 in Subsection 3.1, and apply it to prove Corollary 0.2 in Subsection 3.2

3.1 Quantisation commutes with reduction

The proof of Theorem 0.1 is based on Proposition 1.2, which relates Dirac induction to reduction on $R(K)$ and on $K_d(C_r^*(G))$, and the quantisation commutes with induction principle, Theorem 2.5.

In addition, we will use the fact that Spin^c-quantisation commutes with reduction in the compact case (see Theorem 1.1 from [17]). This result states

that the following diagram commutes for every dominant weight $\lambda = i\xi \in \Lambda^*_+$:

$$
\begin{tikzcd}
\mathrm{CHamPS}(K) \arrow[r, "Q_K"] \arrow[d, "\mathrm{MWR}_\xi"'] & R(K) \arrow[d, "R^\lambda_K"] \\
\mathrm{CHamPS}(\{e\}) \arrow[r, "Q"] & \mathbb{Z}.
\end{tikzcd}
\tag{13}
$$

Here MWR denotes Marsden-Weinstein reduction, including prequantisations and Spin^c-structures.

Combining Proposition 1.2 and Theorem 2.5 with Diagram (13), we obtain the following diagram:

$$
\begin{tikzcd}
\mathrm{CpHamPS}(G) \arrow[r, "Q_G"] & K_d(C^*_r(G)) \arrow[dd, bend left, "R^{\lambda+\rho}_G"] \\
\mathrm{CHamPS}(K) \arrow[u, "\mathrm{pH\text{-}Ind}^G_K"] \arrow[r, "Q_K"] \arrow[d, "\mathrm{MWR}_\xi"'] & R(K) \arrow[u, "\mathrm{D\text{-}Ind}^G_K"'] \arrow[d, "R^\lambda_K"'] \\
\mathrm{CHamPS}(\{e\}) \arrow[r, "Q"] & \mathbb{Z}.
\end{tikzcd}
\tag{14}
$$

Let (N, ν) be a compact, Spin^c-prequantisable Hamiltonian K manifold, and let $(M = G \times_K N, \omega)$ be the induced pre-Hamiltonian G-manifold as in the definition of pre-Hamiltonian induction. By (6), there are integers $m_{\lambda+\rho}$, such that

$$
Q_G(M, \omega) = \sum_{\lambda \in \Lambda^*_+} m_{\lambda+\rho} [\pi_{\lambda+\rho}] \quad \in K_d(C^*_r(G)).
$$

Commutativity of Diagram (14) implies that, for all $\lambda = i\xi \in \Lambda^*_+$,

$$
\begin{aligned}
m_{\lambda+\rho} &= R^{\lambda+\rho}_G\big(Q_G(M, \omega)\big) \\
&= Q(M_\xi, \omega_\xi),
\end{aligned}
$$

as claimed in Theorem 0.1.

3.2 Orbit method for the principal series

Fix an element $\lambda = i\xi \in \Lambda^*_+$. To prove Corollary 0.2, we apply Theorem 0.1 to the case where $N = K \cdot \xi$ is the K-coadjoint orbit through ξ. It remains to prove the fibre bundle structure of the map π (Lemma 3.1), and the fact that the pulled-back form $\pi^*\omega^\lambda$ is the two-form on M^λ induced by the symplectic form on $K \cdot \xi$ as in the pre-Hamiltonian induction procedure (Lemma 3.2). Corollary 0.2 then follows, if we note that the symplectic reduction of $K \cdot \xi$ at an element $\xi' = \lambda'/i$ (for $\lambda' \in \Lambda^*_+$), is a point if $\lambda = \lambda'$, and empty otherwise.

Lemma 3.1. *The map π in (3) defines a fibre bundle over $\mathcal{O}^\lambda$, with fibres diffeomorphic to G_ξ/K_ξ.*

Proof. Fix $g_0 \in G$, and consider the fibre $\pi^{-1}(g_0\xi)$. We first note that

$$\pi^{-1}(g_0\xi) = \{[g_0 a, \xi]; a \in G_\xi\}. \tag{15}$$

Indeed, for any $g \in G$ and $k \in K$, one has $[g, k\xi] = [g_0 a, \xi]$ for $a := g_0^{-1} g k \in G_\xi$.

Next, consider the map

$$\tau : \pi^{-1}(g_0\xi) \to G_\xi / K_\xi,$$

defined by

$$\tau[g_0 a, \xi] = [a],$$

for $a \in G_\xi$. This map is well-defined, because if $[g_0 a, \xi] = [g_0 a', \xi]$, for $a, a' \in G_\xi$, there is a $k \in K_\xi$ such that $g_0 a' = g_0 a k^{-1}$, so $[a'] = [a]$. Surjectivity of τ is immediate from (15), and injectivity follows directly from the definitions. The map τ is smooth, and it inverse is induced by $a \mapsto [ga, \xi]$, and hence smooth as well. □

Lemma 3.2. *Let ω be the closed two-form on M^λ induced from the standard symplectic form ν on $K \cdot \xi$ as in* (10)*. Then $\omega = \pi^* \omega^\lambda$.*

Proof. Let $X, X' \in \mathfrak{k}$ and $Y, Y' \in \mathfrak{p}$ be given. For $k \in K$, a tangent vector in $T_{[e,k\xi]} M^\lambda$ has the form $Y + (X + \mathfrak{k})$. We compute

$$\begin{aligned}
\omega_{[e,k\xi]}\big(Y + (X + \mathfrak{k}_\xi), Y' + (X' + \mathfrak{k}_\xi)\big) &= \nu_{k\xi}(X + \mathfrak{k}_\xi, X' + \mathfrak{k}_\xi) - \langle k \cdot \xi, [Y, Y'] \rangle \\
&= \langle k\xi, [X, X'] + [Y, Y'] \rangle \\
&= \langle k\xi, [X + Y, X' + Y'] \rangle.
\end{aligned}$$

Here we have used the fact that $[X, Y']$ and $[Y, X']$ are in $\mathfrak{p}$, and $k\xi \in \mathfrak{k}^*$ annihilates $\mathfrak{p}$.

On the other hand, one has

$$T_{[e,k\xi]}\pi\big(Y + (X + \mathfrak{k}_\xi)\big) = X + Y + \mathfrak{k}_\xi$$

(and similarly for X' and Y'), so that

$$\pi^* \omega^\lambda_{[e,k\xi]}\big(Y + (X + \mathfrak{k}_\xi), Y' + (X' + \mathfrak{k}_\xi)\big) = \langle k\xi, [X + Y, X' + Y'] \rangle.$$

Therefore, the forms ω and $\pi^*\omega^\lambda$ are equal at points of the form $[e, k\xi]$, and hence on all of M by G-invariance. □

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